2.3 Composition + Matrix Multiplication

$\boxed{\text{Ex}}$ Consider $S: \mathbb{R}^2 \to \mathbb{R}^2$

$R: \mathbb{R}^2 \to \mathbb{R}^3$

given by matrices

$$[S] = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{tikzcd}
\vec{z} = R(\vec{y}) \in \mathbb{R}^3 & \\
& \vec{y} = S(\vec{x}) \in \mathbb{R}^2 \arrow[ul, "R"'] \\
\vec{x} \in \mathbb{R}^2 \arrow[uu, "R \circ S \text{ ``R of S''}"] \arrow[ur] &
\end{tikzcd}$$

$$R \circ S(\vec{x}) = R[S(\vec{x})]$$

$$= \begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right)$$

$$= \begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} 5 \\ 15 \end{pmatrix} = \begin{pmatrix} 30 \\ 35 \\ -20 \end{pmatrix}$$

$$S \circ R(\vec{x}) = S[R(\vec{x})]$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \left(\begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right)$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}$$

Mismatch Dimensions!!!

Theorem - Suppose $S: \mathbb{R}^n \to \mathbb{R}^p$, $R: \mathbb{R}^p \to \mathbb{R}^m$ are linear transformations. Then so is $R \circ S: \mathbb{R}^n \to \mathbb{R}^m$. In fact, $R \circ S$ has matrix

$$[R \circ S] = \underbrace{[R][S]}_{\text{name}} = \begin{bmatrix} [R]\vec{v}_1 & \dots & [R]\vec{v}_n \end{bmatrix}$$

Proof - Write the matrix for S column-wise $[S] = [\vec{v}_1 \ \dots \ \vec{v}_n]$

So if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

Then $S(\vec{x}) = [S]\vec{x} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$

Hence

$$
\begin{aligned}
R[S(\vec{x})] = [R]\, S(\vec{x}) &= [R](x_1\vec{v}_1 + \dots \ x_n\vec{v}_n) \\
&= x_1 [R]\,\vec{v}_1 + \dots + x_n [R]\,\vec{v}_n \\
&= \Big[[R]\,\vec{v}_1 \ \dots \ [R]\,\vec{v}_n \Big] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

In Example

$$
[R][S] = \begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ 1 & 12 \\ 2 & -6 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 4 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}
$$

So $[R \circ S] \begin{bmatrix} -1 \\ 3 \end{bmatrix} =$

$$
\begin{bmatrix} -6 & 8 \\ 1 & 12 \\ 2 & -6 \end{bmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 35 \\ -20 \end{pmatrix}
$$

Example

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

MATRIX MULTIPLICATION IS NOT COMMUTATIVE

Properties of matrix multiplication

Let A, B, C be matrices and $\lambda \in \mathbb{R}$ be a scaler. Then all of the following are true (assuming compatible sizes)

1. $0A = A0$ (0 is the zero matrix)
2. $I_{m\times m} A_{m\times n} = A_{m\times n} I_{n\times n} = A$
3. $A(B+C) = AB + AC$
4. $(A+B)C = AC + BC$
5. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
6. $A(BC) = (AB)C$

Prove $IA = A$

$IA = I[\vec{v}_1 \dots \vec{v}_n] = [Iv_1 \dots Iv_n] = [v_1 \dots v_n] = A$

Prove $AI = A$ (could do this the same as above as well)

- Let $R: \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation with matrix $[R] = A$
- Let $id: \mathbb{R}^n \to \mathbb{R}^n$ be linear transformation with matrix I $id(\vec{x}) = I\vec{x} = \vec{x}$

Then OTOH

$R \circ id(\vec{x}) = R(\vec{x}) = A(\vec{x})$

But OTOH

$[R \circ id] = [R][id] = AI$

So

$[R \circ id][\vec{x}] = AI\vec{x}$

$\therefore A = AI$